CHAPTER 4

The Bernoulli Equation and Pressure Variation

CHAPTER ROAD MAP
This chapter describes flowing fluids, introduces the Bernoulli equation, and describes pressure variations in flowing fluids.

Learning Objectives: Students will be able to
• Describe streamlines, streaklines, and pathlines. Explain how these ideas differ (§4.1).
• Describe velocity and the velocity field (§4.2).
• Describe the Eulerian and Lagrangian Approaches (§4.2).
• Describe flowing fluids using the concepts introduced in section §4.3.
• Define acceleration. Sketch the direction of the acceleration vector of a fluid particle. Define local acceleration and convective acceleration. (§4.4)
• Apply Euler’s equation to describe pressure variations (§4.5).
• Apply the Bernoulli equation along a streamline (§4.6).
• Define static pressure and kinetic pressure. Explain how to measure velocity using a Pitot-static tube (§4.7).
• Define the rate of rotation and vorticity. Define an irrotational flow. (§4.8)
• Apply the Bernoulli equation in an irrotational flow (§4.9).
• Define the pressure coefficient. Sketch the pressure variation for flow around a circular cylinder (§4.10).
• Calculate the pressure variation in a rotating flow (§4.11).

4.1

Describing Streamlines, Streaklines, and Pathlines

To visualize and describe flowing fluids, engineers use the streamline, streakline, and pathline. Hence, these topics are introduced in this section.
Pathline, Streamlines, and Streaklines

- To visualize flow engineers use the streamline, streakline and the pathline.
- The streamline is a curve that is everywhere tangent to the local velocity vector.
- The configuration of streamlines in a flow field is called the flow pattern.
- The pathline is the line (straight or curved) that a particle follows.
- A streakline is the line produced by a dye or other marker fluid introduced at a point.
- In steady flow, pathlines, streaklines, and streamlines are coincident (i.e., the same) if they share a common point.
- In unsteady flow, pathlines, streaklines, and streamlines are not coincident (i.e., not the same).

Velocity and Velocity Field

- In a flowing fluid, velocity is defined as the speed and direction of travel of a fluid particle.
- A velocity field is a mathematical or graphical description that shows the velocity at each spatial location within a flow.

Eulerian and Lagrangian Descriptions

There are two ways to describe motion (Lagrangian and Eulerian).

- In the Lagrangian approach, the engineer identifies a specified collection of matter and describes its motion. For example, when an engineer is describing the motion of a fluid particle this is a Lagrangian-based description.
- In the Eulerian approach, the engineer identifies a region in space and describes the motion of matter that is passing by in terms of what is happening at various spatial locations. For example, the velocity field is a Lagrangian-based concept.

- The Eulerian approach uses fields. A field is a mathematical or graphical description that shows how a variable is distributed spatially. A field can be a scalar field or a vector field.
- The Eulerian approach uses the divergence, gradient, and curl operators.
- The Eulerian approach uses complicated mathematics (e.g., partial derivatives) than the Lagrangian approach.

Describing Flow

Engineers describe flowing fluids using the ideas summarized in Table 4.4.

<table>
<thead>
<tr>
<th>Description</th>
<th>Key Knowledge</th>
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<tbody>
<tr>
<td>Engineers classify flows as uniform or nonuniform.</td>
<td>• Uniform flow means the velocity of each fluid particle in a flow is constant (both speed and direction of travel). Uniform flow requires steady flow and rectilinear streamlines (straight and parallel). • Nonuniform flow means the velocity of one or more fluid particles is changing (speed changes, direction of travel changes or both change).</td>
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PROBLEMS

Table 4.4  How Engineers Describe Flowing Fluids

<table>
<thead>
<tr>
<th>Description</th>
<th>Key Knowledge</th>
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<tbody>
<tr>
<td>Engineers classify flows as steady or unsteady.</td>
<td>• Steady flow means the velocity is constant with respect to time at every point in space.</td>
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<td></td>
<td>• Unsteady flow means the velocity is changing with time at some or all points in space.</td>
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<td></td>
<td>• Engineers often idealize unsteady flows as steady flow. Example: A draining tank of water is commonly assumed to be a steady flow.</td>
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<td>Engineers classify flows as laminar or turbulent.</td>
<td>• Laminar flow involves flow in smooth layers (laminae) with low levels of mixing between layers.</td>
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<td>• Turbulent flow involves flow that is dominated by eddies of various size. Flow is chaotic, unsteady, 3D. High levels of mixing. Occasionally, engineers describe a flow as transitional. This means that the flow is changing from a laminar flow to a turbulent flow.</td>
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<tr>
<td>Engineers classify flows as 1D, 2D, or 3D.</td>
<td>• One dimensional (1D) flow means the velocity depends only on one spatial variable. E.g. Velocity depends on radius ( r ) only.</td>
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<tr>
<td></td>
<td>• Three dimensional (3D) flow means the velocity depends only on three spatial variable. E.g. Velocity depends on three position coordinates: ( \mathbf{V} = \mathbf{V}(x, y, z) ).</td>
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<td>Engineers classify flows as viscous flow or inviscid flow.</td>
<td>• In a viscous flow, the forces associated with viscous shear stresses are significant. Thus, viscous terms are included when solving the equations of motion.</td>
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<tr>
<td></td>
<td>• In an inviscid flow, the forces associated with viscous shear stresses are insignificant. Thus, viscous terms are neglected when solving the equations of motion. The fluid behaves as if its viscosity were zero.</td>
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<td>Engineers describe flows by describing an inviscid flow region, a boundary layer, and a wake.</td>
<td>• In the inviscid flow region, the streamlines are smooth and the flow can be analyzed with Euler’s equation.</td>
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<td>• The boundary layer is a thin region of fluid next to wall. Viscous effects are significant in the boundary layer.</td>
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<td>• The wake is the region of separated flow behind a body.</td>
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<tr>
<td>Engineers describe flows as separated or attached.</td>
<td>• Flow separation is when fluid particles move away from the wall.</td>
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<td>• Attached flow is when fluid particles are moving along a wall or boundary.</td>
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<td>• The region of separated flow inside a pipe or duct is often called a recirculation zone.</td>
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Acceleration

• Acceleration is a property of a fluid particle that characterizes
  • The change in speed of the particle
  • The change in direction of travel of the particle
• Acceleration is defined mathematically as the derivative of the velocity vector.
• Acceleration of a fluid particle can be described qualitatively. Guidelines.
  • If a particle is traveling on a curved streamline, there will be a component of acceleration that is normal to the streamline and directed inwards towards the center of curvature.
  • If the particle is changing speed, there will be a component of acceleration that is tangent to the streamline.
• In an Eulerian representation of acceleration,
  • Terms that involve derivatives with respect to time are local acceleration terms.
  • All other terms are convective acceleration terms. Most of these terms involve derivatives with respect to position.
Euler’s Equation

- *Euler’s equation*, which Newton’s second law for a fluid particle, applies to inviscid flow of an incompressible fluid.
- Euler’s equation can be written as a *vector equation*:
  \[-\nabla p_z = \rho a\]

- This vector form can be also be written as a *scalar equation* in an arbitrary \( \ell \) direction.
  \[-\frac{\partial}{\partial \ell}(p + \gamma z) = -\left(\frac{\partial p_z}{\partial \ell}\right) = \rho a_\ell\]

*Physics of Euler’s equation*: The gradient of the piezometric pressure is colinear with the acceleration of the fluid particle and opposite in direction. This reveals how pressure varies:
- When streamlines are curved, pressure will increase outward from the center of curvature.
- When a streamline is rectilinear and a particle on the streamline is changing speed, then the pressure will change in a direction tangent to the streamline which increasing pressure in the direction opposite of the acceleration vector.
- When streamlines are rectilinear, pressure variation normal to the streamlines is hydrostatic.

The Bernoulli Equation

- The *Bernoulli equation* is conservation of energy applied to a fluid particle. It is derived by integrating Euler’s equation for steady, inviscid, and constant density flow.
- For the assumptions just stated, The Bernoulli equation is applied between any two points on the same streamline.
- The Bernoulli equations has two forms:
  - **Head Form**: \( p/\gamma + z + V^2/(2g) = \text{constant} \)
  - **Pressure Form**: \( p + \rho gz + (\rho V^2)/2 = \text{constant} \)
- There are two equivalent ways to describe the physics of the Bernoulli equation
  - When speed increases, then piezometric pressure decreases (along a streamline).
  - The total head (velocity head plus piezometric head) is constant along a streamline. This means that energy is conserved as a fluid particle moves along a streamline.

Measuring Velocity and Pressure

- When pressure is measured at a *pressure tap* on the wall of a pipe, this provides a measurement of static pressure. This same measurement can also be used to determine pressure head or piezometric head.
- *Static pressure* is defined as the pressure in a flowing fluid. Static pressure must be measured in a way that does not change the value of the measured pressure.
- *Kinetic pressure* is \( p + (\rho V^2)/2 \).
- A *stagnation tube* provides a measurement of kinetic pressure.
- The *Pitot-static* tube, provides a method to measure both static pressure and kinetic pressure at a point in a flowing fluid. Thus, this instrument provides a way to measure fluid velocity.
Fluid Rotation, Vorticity, and Irrotational Flow

- Rate of rotation $\Omega$
  - is a property of a fluid particle that describes how fast the particle is rotating.
  - is defined by placing two perpendicular lines on a fluid particle and then averaging the rotational rate of these lines.
  - is a vector quantity with the direction of the vector given by the right hand rule.
- A common way to describe rotation is to use the vorticity vector $\omega$ which is twice the rotation vector: $\omega = 2\Omega$
- In Cartesian coordinates, the vorticity is given by

$$\omega = \left( \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z} \right) \mathbf{i} + \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \mathbf{j} + \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \mathbf{k}$$

- An irrotational flow is one in which vorticity is everywhere zero.
- When applying the Bernoulli equation for irrotational flow, one can select points 1 and 2 at any locations, not just along a streamline.

Describing the Pressure Field

- The pressure field is often described using a $\pi$-group called the pressure coefficient.
- The pressure gradient near a body is related to flow separation
  - An adverse pressure gradient is associated with flow separation.
  - A positive pressure gradient is associated with attached flow.
- The pressure field for flow over a circular cylinder is a paradigm for understanding external flows. The pressure along the front of the cylinder is high and the pressure in the wake is low.
- When flow is rotating as a solid body, the pressure field $p$ can be described using

$$p + \gamma z - \rho \frac{\omega^2 r^2}{2} = C$$

where $\omega$ is the rotational speed, and $r$ is the distance from the axis of rotation to the point in the field.

Describing the Pressure Field (Summary)

Pressure variations in a flowing fluid are associated with three phenomenon:
- **Weight.** Due to the weight of a fluid, pressure increases with increasing depth (i.e. decreasing elevation). This topic is presented in Chapter 3 (Hydrostatics)
- ** Acceleration.** When fluid particles are accelerating, there are usually pressure variations associated with the acceleration. In inviscid flow, the gradient of the pressure field is aligned in a direction opposite of the acceleration vector.
- **Viscous Effects.** When viscous effects are significant, there can be associated pressure changes. For example, there are pressure drops associated with flows in horizontal pipes and ducts. This topic is presented in Chapter 10 (Conduit Flow).

References